MTH 1420, SPRING 2012
DR. GRAHAM-SQUIRE

LAB 10: WHEN DOES A FUNCTION HAVE A TAYLOR SERIES?

Name: ____

1. Instructions

Your group should write up and turn in one completed lab at the start of the next lab period. You can use this sheet as a cover sheet for the lab you turn in. **Each member of the group should write up at least part of the lab**, but you should check each other's work since everyone in the group gets the same score.

2. Introduction

In class we have looked at certain functions and their Taylor series, and we have assumed that those functions have a Taylor series. However, not all functions will have a Taylor series representation, so in this lab we will discuss what needs to happen in order to show that a function does, in fact, have a Taylor series representation at a given point.

3. Taylor polynomials and remainders

Suppose you are given a function f and it has derivatives of all orders (that is, $f^{(n)}(x)$ is defined for all positive integers n) for a certain values $x = a$. When is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n?$$

First, let's look at the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1}(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}+\ldots$$

Definition 1. Let

$$f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = T_n(x)$$

be the nth degree Taylor polynomial of f at a and

$$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \dots$$

be the remainder at n of the Taylor series.

Exercise 2. Write the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ in terms of $T_n(x)$ and $R_n(x)$.

4. When a Taylor series equals a function

We have the following Theorem:

Theorem 4.1. *If $\lim_{n\to\infty} R_n(x) = 0$, then the function f is equal to its Taylor series within the interval of convergence.*

Unfortunately, calculating that limit can be rather difficult, so we usually use an approximation to make it easier. Namely:

Theorem 4.2. *(Taylor's inequality) If $f^{n+1}(x) \leq M$ for $|x-a| < d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

Taylor's inequality is helpful, because if it holds then we know that $\lim_{n\to\infty} R_n(x) = 0$.

Exercise 3. Calculate $\lim_{n\to\infty} \frac{M}{(n+1)!}|x-a|^{n+1}$ and use it (and Taylor's inequality) to explain why $\lim_{n\to\infty} R_n(x) = 0$. (It may be hard to prove that limit mathematically, but you can use this trick to do it: Consider the series $\sum_{n=0}^{\infty} \frac{M}{(n+1)!}|x-a|^{n+1}$. Use the ratio test to show that the series converges. If a series converges, then the terms of the series must go to zero as n goes to infinity, thus proving what you want.)

We will now use this to prove that the Maclaurin series $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ we found in class actually represents that function. What we need to prove is that $\lim_{n\to\infty} R_n(x) = 0$. We will use Taylor's inequality to do it, so our first step is to find an appropriate value for M.

Exercise 4. Find a number M such that $|f^{(n)}(x)| \leq M$ for all values of x and all positive integers n, where $f(x) = \cos x$. (Hint: start taking derivatives of $\cos x$ and see if you can find an M value that will be an upper bound for all of them. You should choose the smallest M possible.)

Exercise 5. Using Taylor's inequality we see that

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Plug in your value for M and explain why $\lim_{n\to\infty} \frac{M}{(n+1)!}|x-a|^{n+1} = 0$.

According to Theorem **??**, you have just shown that $\cos x$ is equal to its Taylor series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ within the interval of convergence, which happens to be infinity.

5. A function that does not have a Maclaurin series

We will do some brief exploration of a function that does not have a Maclaurin series. Namely, consider the function

$$f(x) = \begin{cases} e^{(-1/x^2)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Exercise 6. If $f(x)$ had a Maclaurin series, what would it look like (for all values of x not equal to zero, at least). (Hint: Use the Maclaurin series for e^x and substitute something in for x.)

Exercise 7. Sketch a graph of $f(x)$ (Use a graphing calculator or sage and then copy it to your page).

Exercise 8. Use the definition of continuity ($\lim_{x\to a} f(x) = f(a)$) to explain/show that $f(x)$ is continuous at $x = 0$ (Note: the graph is not enough explanation).

Exercise 9. Find $f'(x)$ and explain why it is defined. (Note: to do this, you must explain why $\lim_{x\to 0} f'(x) = 0$. This may be difficult to prove, but you should at least justify it by plugging some x-values close to zero into $f'(x)$ and showing that they will in fact go to zero).